Declarative Pearl:

Deriving Monadic Quicksort

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Monadic Specification of Sorting

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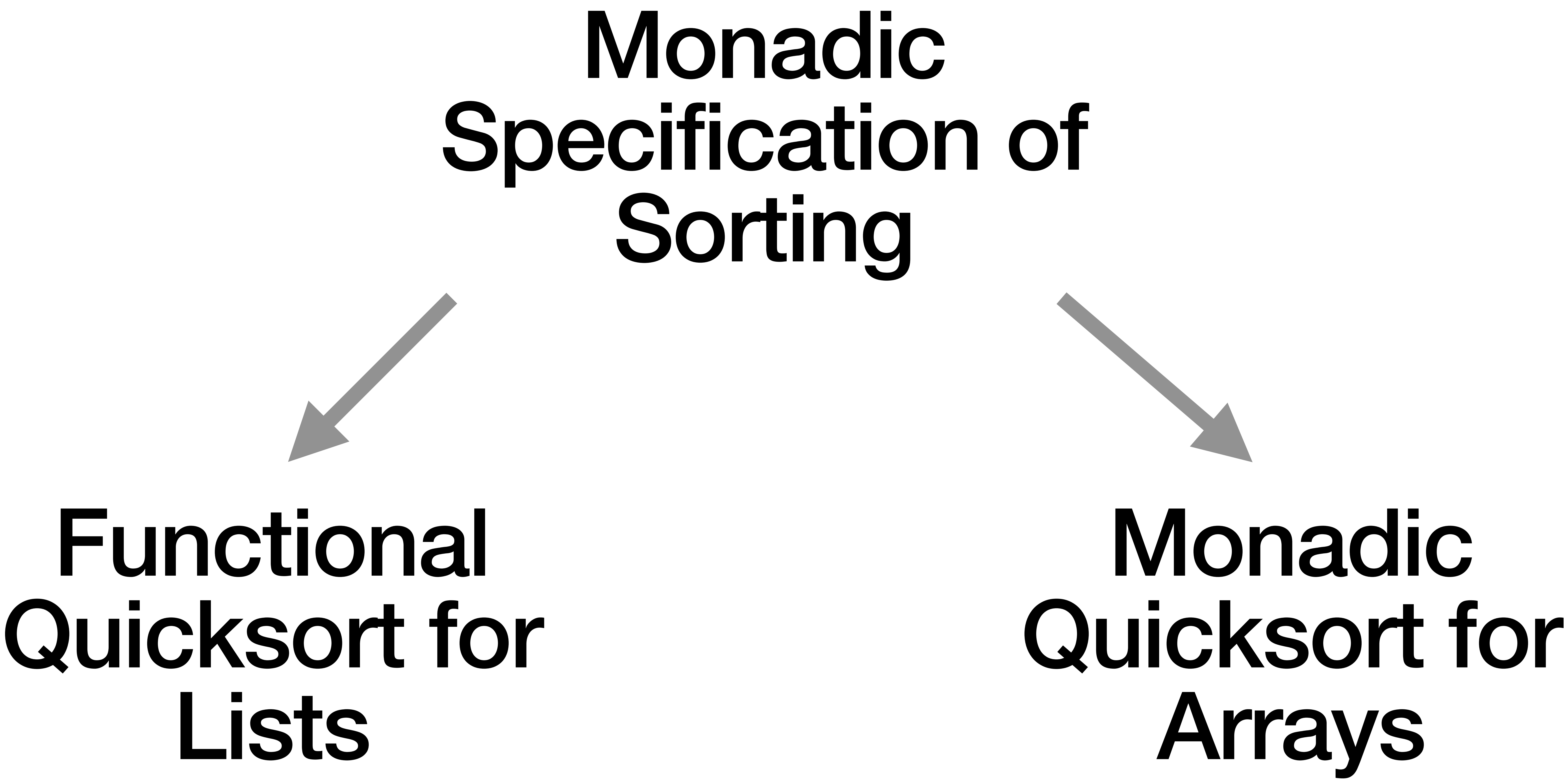
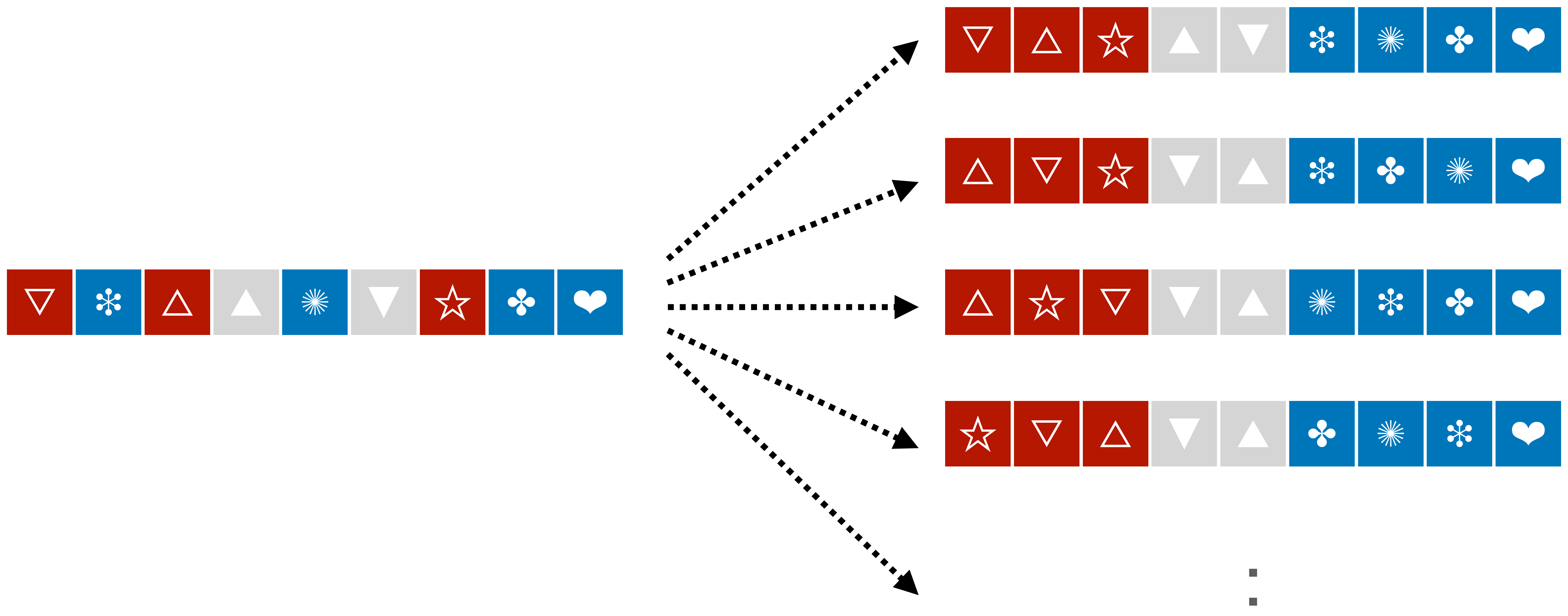
Functional Quicksort for Lists


Monadic Specification of Sorting
Functional Quicksort for Lists
Monadic Quicksort for Arrays

Program Derivation

$$
\begin{array}{ll}
 & \text{problem specification} \\
= & \quad \{ \text{ reason 1 } \} \\
 & \text{expr 1} \\
= & \quad \{ \text{ reason 2 } \} \\
 & \text{expr 2} \\
 & \quad \vdots \\
= & \quad \{ \text{ reason n } \} \\
 & \text{implementation.}
\end{array}
$$

Non-determinism



Some sorting algorithms can be more efficient if we are not over specific about order of items with equal keys.

Relational Program Derivation

relational specification

$\supseteq$ { reason 1 }

expr 1

$=$ { reason 2 }

expr 2

⋮

$\supseteq$ { reason n }

functional implementation.

$R \supseteq S$: relational inclusion;
whatever S does is allowed by R.

Relational Program Derivation

Developed around late 80's - 90's.

I still love it!

"Bizarre, too complex."

Point-free. Harder to apply usual techniques such as pattern matching, induction on the arguments, etc.

Pointwise notation confusing when applying a function to a non-deterministic value.

Monadic Program Derivation

$$
\begin{array}{ll}
 & \textbf{Monadic-Spec}\ (x{:}xs) \\
\supseteq & \quad \{\ \text{reason } 1\ \} \\
 & \text{expr } 1 \\
= & \quad \{\ \text{reason } 2\ \} \\
 & \text{expr } 2 \\
 & \quad \vdots \\
\supseteq & \quad \{\ \text{reason } n\ \} \\
 & \textbf{return}\ (\text{implementation}\ (x{:}xs))\ .
\end{array}
$$

$R \supseteq S$: to be defined later!

Monadic Program Derivation

Non-determinism represented by monads.

One can apply usual functional program derivation techniques --- e.g. structural induction on input data.

Possibility of incorporating other effects.

Monad

operators

$$\text{return} :: a \rightarrow m\ a$$
$$(\gg\!=) :: m\ a \rightarrow (a \rightarrow m\ b) \rightarrow m\ b$$

monad laws

$$\text{return}\ x \gg\!= f \quad = \quad f\ x$$
$$f \gg\!= \text{return} \ = \ f$$
$$(m \gg\!= f) \gg\!= g \ = m \gg\!= (\lambda\ x \rightarrow f\ x \gg\!= g)$$

Non-determinism Monad

operators

(▯) :: m a → m a → m a

∅ :: m a

Non-determinism Monad

operators

$$(\square) \;::\; m\ a \to m\ a \to m\ a$$

$$\varnothing \;::\; m\ a$$

monoid

$$(m \;\square\; n) \;\square\; k \;=\; m \;\square\; (n \;\square\; k)$$

$$m \;\square\; \varnothing = m = \varnothing \;\square\; m$$

idempotence

$$m \;\square\; m = m$$

commutativity

$$m \;\square\; n = n \;\square\; m$$

Insertion & Permutation

```
insert :: a → [a] → m [a]
insert y [ ]      = return [y]
insert y (x:xs) = return (y:x:xs) ▯
           (x:) ⟨$⟩ insert y xs
```

Insertion & Permutation

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insert :: a → [a] → m [a]
insert y [ ]      = return [y]
insert y (x:xs) = return (y:x:xs) ▯
            (x:) ⟨$⟩ insert y xs

f ⟨$⟩ m  =  m ≫= (λ x → return (f x))
```

Insertion & Permutation

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          (x:) ⟨$⟩ insert y xs

f ⟨$⟩ m  =  m ≫= (λ x → return (f x))

perm :: [a] → m [a]
perm [ ]      = return [ ]
perm (x:xs) = perm xs ≫=  insert x
```

Sorting

$$\text{slowsort } xs = \text{perm } xs \gg\!= \text{filt ordered}$$

$$\begin{aligned}\text{filt } p\ x &= \textbf{if } p\ x \textbf{ then } \text{return } x \textbf{ else } \varnothing \\ &= \text{guard } (p\ x) >> \text{return } x\end{aligned}$$

$$\text{guard } b = \textbf{if } b \textbf{ then } \text{return } (\,) \textbf{ else } \varnothing$$

Program Refinement

$$m_1 \subseteq m_2 \;\equiv\; (m_1 \,\square\, m_2 = m_2)$$

Divide & Conquer

```
perm [ ] = {[ ]}
perm (x:xs) =
      split xs ≫= λ (ys,zs) →
      perm ys ≫= λ ys' →
      perm zs ≫= λ zs'→
      liftM2 (⧺[x]⧺) ys' zs'
```

splits : all the ways to split a list into two.

Deriving Quicksort

$$\begin{aligned}
\text{slowsort } [\,] &= \{[\,]\} \\
\text{slowsort } (x{:}xs) &= \text{perm } (x{:}xs) \gg\!= \text{filt ordered} \\
&= \,?
\end{aligned}$$

```
slowsort (x:xs) =
    perm (x:xs) >>= filt ordered
```

```
slowsort (x:xs) =
    split xs >>= λ (ys,zs) →
    perm ys >>= λ ys' →
    perm zs >>= λ zs' →
    filt ordered (ys' ++ [x] ++ zs')
```

```
slowsort (x:xs) =
    split xs >>= λ (ys,zs) →
    perm ys >>= λ ys' →
    perm zs >>= λ zs' →
    guard (ordered ys' ∧ ordered zs' ∧
           all (≤x) ys' ∧ all (≥x) zs')) >>
    return (ys' ++ [x] ++ zs')
```

```
slowsort (x:xs) =
    split xs >>= λ (ys,zs) →
    guard (all (≤x) ys' ∧ all (≥x) zs')) >>
    (perm ys >>= filt sorted)>>= λ ys' →
    (perm zs >>= filt sorted)>>= λ zs'→
    return (ys' ++ [x] ++ zs')
```

```
slowsort (x:xs) =
    split xs >>= λ (ys,zs) →
    guard (all (≤x) ys' ∧ all (≥x) zs')) >>
    slowsort ys >>= λ ys' →
    slowsort zs >>= λ zs'→
    return (ys' ++ [x] ++ zs')
```

```
slowsort (x:xs) =
  split xs >>= λ (ys,zs) →
  guard (all (≤x) ys' ∧ all (≥x) zs')) >>
  slowsort ys >>= λ ys' →            ⊇ partition x xs
  slowsort zs >>= λ zs'→
  return (ys' ++ [x] ++ zs')
```

Quicksort for Lists

slowsort xs ⊇ return (quicksort xs)

```
quicksort [ ]      = [ ]
quicksort (x:xs) =
   let (ys, zs) = partition x xs
   in quicksort ys ++ [ x ] ++ quicksort zs
```

Commuting Guards

Definition: m and n commute if

$$m \gg\!= \lambda\, x \to n \gg\!= \lambda\, y \to f\, x\, y \quad = \quad n \gg\!= \lambda\, y \to m \gg\!= \lambda\, x \to f\, x\, y$$

Theorem: guard commutes with other terms.

To prove the commutativity we need:

left zero $\varnothing \gg= f = \varnothing$

left distr. $(m_1 \square m_2) \gg= f = (m_1 \gg= f) \square (m_2 \gg= f)$

right zero $m \gg \varnothing = \varnothing$

right distr. $m \gg= (\lambda\, x \to f_1\, x \square f_2\, x) = (m \gg= f_1) \square (m \gg= f_2)$

Arrays

types

Idx -- index to a global array
e -- type of elements in the array

operators

read :: Idx → m e
write :: Idx → e → m ( )

Arrays

types

Idx -- index to a global array
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operators

```
read  :: Idx → m e
write :: Idx → e → m ()
```

induced operators

```
readList  :: Idx → Nat → m [e]
writeList :: Idx → [e] → m ()
swap      :: Idx → Idx → m ()
```

Imperative Quicksort

type $\text{iqsort} :: \text{Idx} \to \text{Nat} \to m\,()$

specification $\text{writeList}\ i\ xs \gg \text{iqsort}\ i\ (\#xs) \subseteq$
$\text{slowsort}\ xs \gg\!= \text{writeList}\ i$

#xs: length of xs

Imperative Quicksort

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Imperative Quicksort

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specification $\underline{\text{writeList } i \text{ } xs} \gg \underline{\text{iqsort } i \text{ } (\#xs)} \subseteq$
$\text{slowsort } xs \gg\!= \text{writeList } i$

Imperative Quicksort

type $\quad$ iqsort :: Idx → Nat → m ( )

specification $\quad$ writeList i xs ≫ iqsort i (#xs) ⊆
slowsort xs >>= writeList i

Imperative Quicksort

type iqsort :: Idx → Nat → m ( )

specification

precondition

$\boxed{\text{writeList } i \; xs} \gg \text{iqsort } i \; (\#xs) \subseteq$

$\text{slowsort } xs \gg\!= \text{writeList } i$

Imperative Quicksort

type $\text{iqsort} :: \text{Idx} \to \text{Nat} \to m\ (\,)$

specification

precondition

$$\boxed{\text{writeList } i \ xs} \gg \text{iqsort } i\ (\#xs) \subseteq$$
$$\boxed{\text{slowsort } xs \gg\!= \text{writeList } i}$$

postcondition

Imperative Quicksort

type $\quad$ iqsort :: Idx → Nat → m ( )

specification

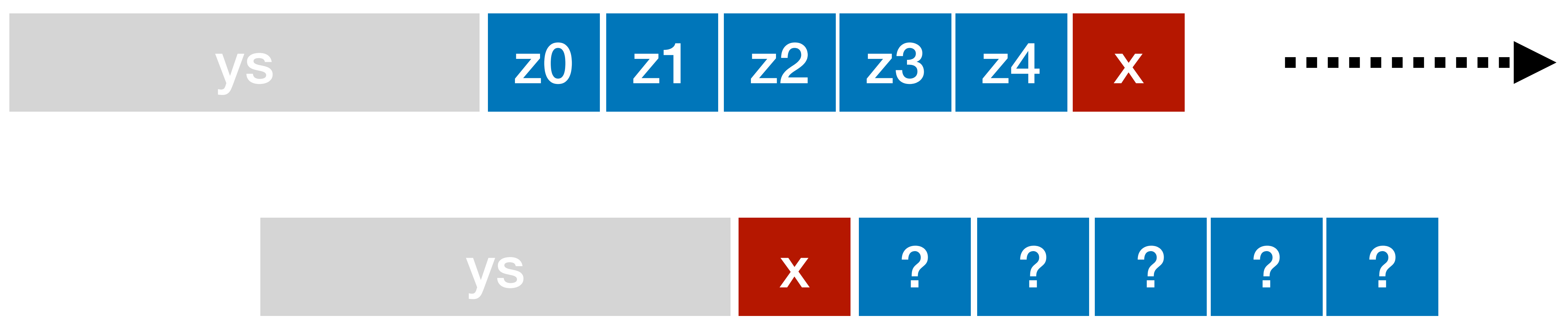
precondition $\qquad$ code to derive

$$\boxed{\text{writeList } i\ xs} \gg \boxed{\text{iqsort } i\ (\#xs)} \subseteq$$
$$\boxed{\text{slowsort } xs \gg\!= \text{writeList } i}$$

postcondition

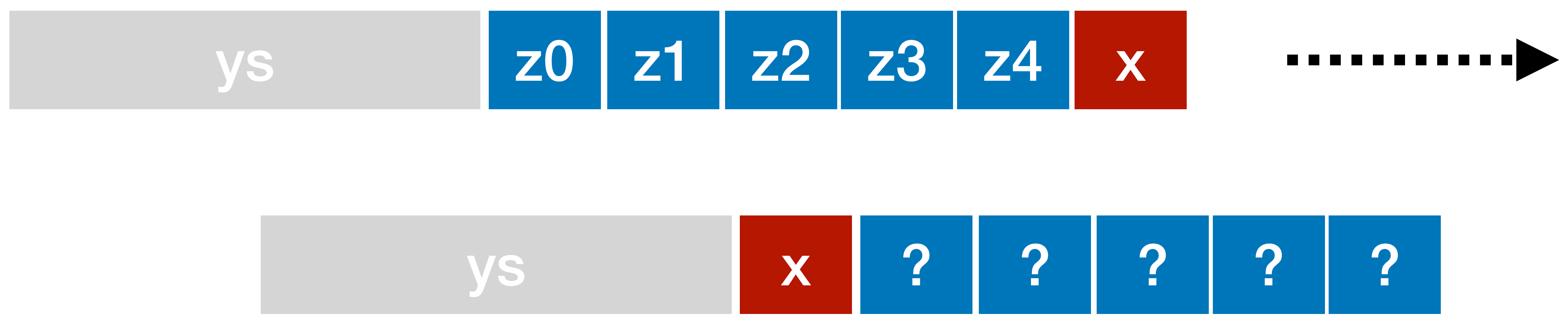
Constructing Imperative Programs

$$\begin{array}{l} \mathit{writeList}\ i\ (ys \mathbin{+\!\!+} zs \mathbin{+\!\!+} [x]) \gg \\ \quad {\color{red}??} \subseteq \\ \mathit{perm}\ zs \gg\!= \lambda\, zs' \rightarrow \\ \quad \mathit{writeList}\ i\ (ys \mathbin{+\!\!+} [x] \mathbin{+\!\!+} zs') \end{array}$$



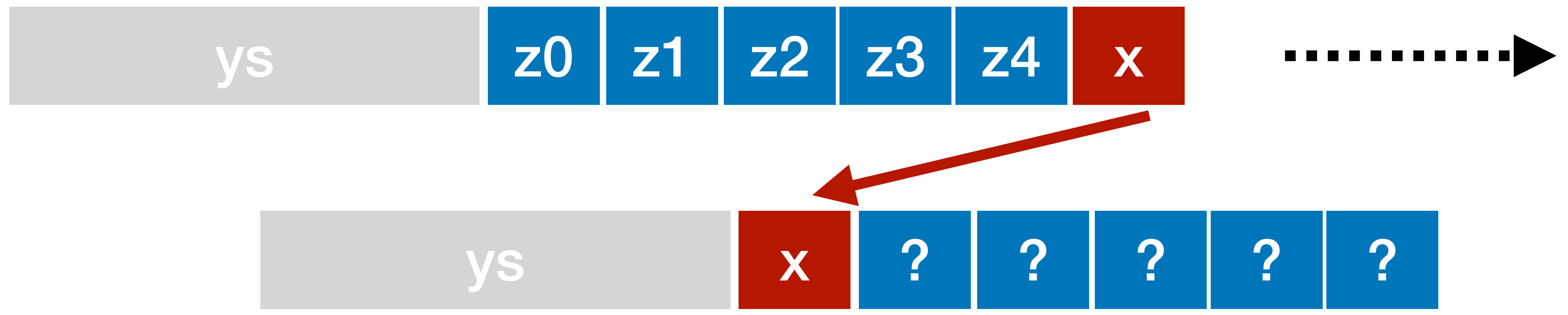
Constructing Imperative Programs

$$\mathit{writeList}\ i\ (\underline{ys \mathbin{+\!\!+} zs \mathbin{+\!\!+} [x])} \gg$$
$$\mathbf{??} \subseteq$$
$$\mathit{perm}\ zs \gg\!\!= \lambda\, zs' \rightarrow$$
$$\mathit{writeList}\ i\ (ys \mathbin{+\!\!+} [x] \mathbin{+\!\!+} zs')$$



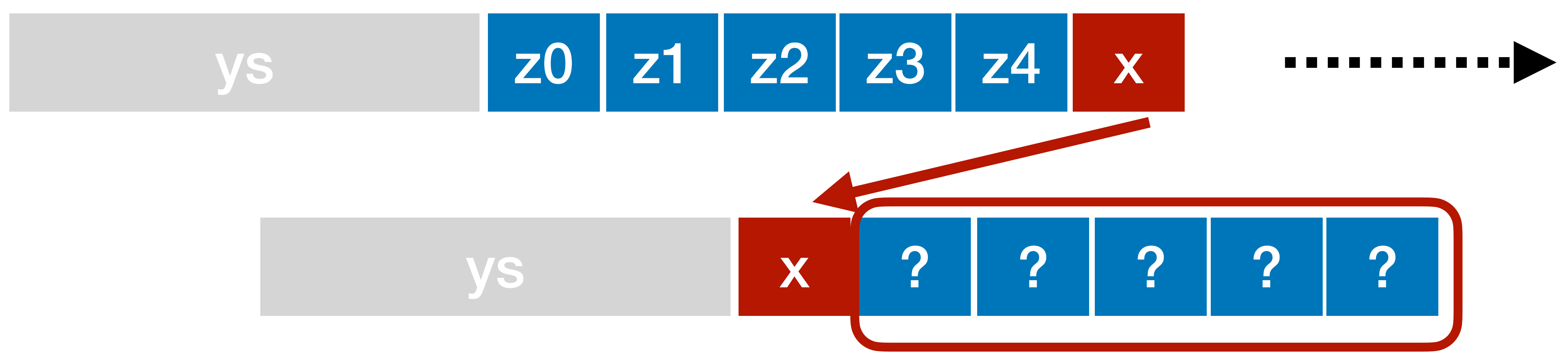
Constructing Imperative Programs

$$\begin{array}{l} \text{writeList } i\ (\underline{ys \mathbin{+\!\!+} zs \mathbin{+\!\!+} [x]}) \gg \\ \quad {\color{red}??} \subseteq \\ \text{perm } zs \gg\!= \lambda\, zs' \to \\ \quad \text{writeList } i\ (\underline{ys \mathbin{+\!\!+} [x] \mathbin{+\!\!+} zs'}) \end{array}$$



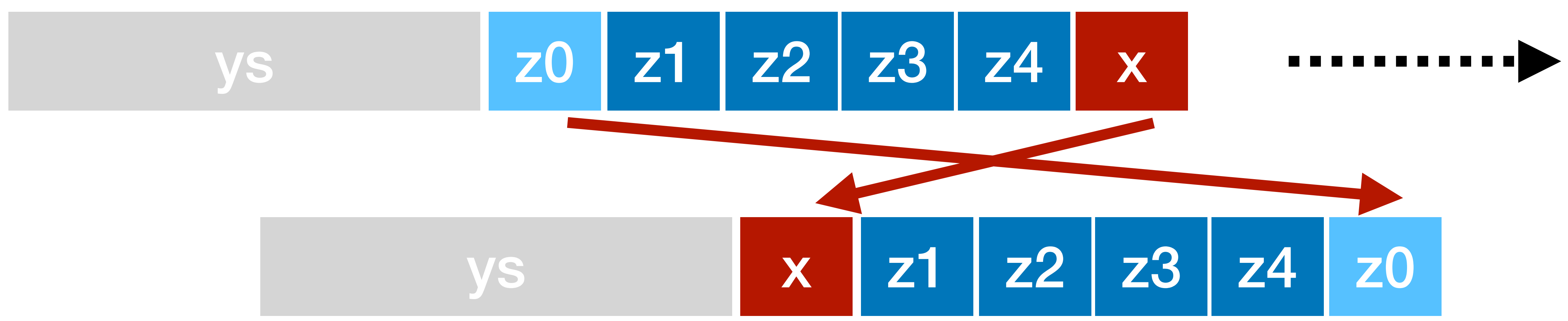
Constructing Imperative Programs

$$
\begin{aligned}
&\mathit{writeList}\ i\ (\underline{ys \mathbin{+\!\!+} zs \mathbin{+\!\!+} [x]}) \gg \\
&\quad \color{red}{??} \subseteq \\
&\mathit{perm}\ zs \gg\!\!= \lambda\, zs' \rightarrow \\
&\quad \mathit{writeList}\ i\ (\underline{ys \mathbin{+\!\!+} [x] \mathbin{+\!\!+} zs'})
\end{aligned}
$$



Constructing Imperative Programs

$$
\begin{array}{l}
\mathit{writeList}\ i\ (ys \mathbin{+\!\!+} zs \mathbin{+\!\!+} [x]) \gg \\
\quad \mathit{swap}\ (i + \#ys)\ (i + \#ys + \#zs) \quad \subseteq \\
\quad \mathit{perm}\ zs \gg\!= \lambda\, zs' \rightarrow \\
\qquad \mathit{writeList}\ i\ (ys \mathbin{+\!\!+} [x] \mathbin{+\!\!+} zs')
\end{array}
$$



Quicksort for Arrays

```
iqsort i 0 = {( )}
iqsort i n =
    read i ≫= λ p →
    ipartition p (i+1) (0,0,n-1) ≫= λ (ny,nz) →
    swap i (i + ny)
    iqsort i ny ≫ iqsort (i+ny+1) nz
```

Conclusions

Monad: a good choice as a calculus for program derivation that involves non-determinism.

Able to apply familiar techniques --- pattern matching, induction on structures or on sizes, etc.

Other effects can be naturally integrated.

Thank you.